JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 27, No. 4, July–August 2004

Analytical Theory of Three-Degree-of-Freedom Aircraft Wing Rock

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Multiple-degree-of-freedom wing-rock cases are usually studied using numerical methods because of the complexity of the problem. Recently, a methodology combining the Multiple Time Scales method, center manifold reduction principle, and bifurcation theory has been developed for obtaining an accurate representation of the two-degree-of-freedom wing-rock dynamics in a parametric form. Such solutions offer an advantage over numerical solutions in that the interdependence of the important parameters affecting the dynamic properties of the system can be easily seen. Also, the fast and slow dynamics of the system are systematically separated, leading to more insight into the complex dynamics of the aircraft. This paper extends the methodology to the more difficult case of wing rock on an aircraft having three rotational degrees of freedom. The excellent accuracy of the analytical solution is then demonstrated by comparison with the numerical result.

Nomenclature

b	=	wing span
C_l	=	coefficient of rolling moment
C_m	=	coefficient of pitching moment
C_{m_q}	=	pitching-moment coefficient caused by pitch rate
$C_{m_{\dot\alpha}}$	=	pitching-moment coefficient caused by rate of change of angle of attack
C_n	=	coefficient of yawing moment
c	=	airfoil chord length
c_D	=	drag coefficient
c_L	=	lift coefficient
g	=	coefficient of gravity
I_{ii}	=	moment of inertia about i axis
I_{ij}	=	product of inertia
$\boldsymbol{i}_{()}$	=	unit vector along the axis indicated in the subscript
p	=	roll rate about X_b axis
q	=	pitch rate about Y_b axis
$\Re(\cdot)$	=	real part of the complex number in the parentheses
r	=	yaw rate about Z_b axis
S	=	wing area
V	=	airspeed
$X_bY_bZ_b$	=	right-handed Cartesian body-fixed axis system
$X_0Y_0Z_0$	=	right-handed Cartesian stability axis system
y	=	coordinate along Y_b axis
α	=	deviation of angle of attack from α_0
α_e	=	effective angle of attack
α_0	=	nominal angle of attack
β	=	angle of sideslip
θ	=	pitch angle
$\tilde\vartheta$	=	$\epsilon\vartheta$
$\tilde\mu$	=	$\epsilon\mu$, roll damping parameter
ρ	=	air density
ϕ	=	roll angle
ψ	=	yaw angle
$\boldsymbol{\omega}$	=	inertial angular velocity
$\dot{(\cdot)}$	=	$= \mathrm{d}/\mathrm{d}t(\cdot)$

Received 22 July 2002; revision received 29 October 2003; accepted for publication 3 November 2003.

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I. Introduction

MODERN fighter aircraft have enhanced performance capabilities for air superiority. Such requirements necessitate flight in nonlinear regimes involving highly complex dynamics, such as wing rock in flight at high angles of attack, which involves sustained lateral oscillations at a constant amplitude and definite frequency. The severity of wing rock is determined by the amplitude of the motion and the period of the oscillation.

Several kinds of wing-rock problems have been treated in the literature, particularly wing rock with only roll degree of freedom,[1–7] with numerical and analytical approaches. For this specific case, the onset of wing rock is influenced by the loss of the roll damping derivative and the amplitude by the nonlinearity in the derivative of the roll moment coefficient with respect to roll rate. Some numerical analyses of wing rock on aircraft with multiple rotational degrees of freedom have also been reported.[8–12]

Recently, an accurate analytical methodology was developed to solve the two-degree-of-freedom wing-rock case.[13,14] This approach is based on the Multiple Time Scales (MTS) method[15,16] and utilizes the center manifold reduction technique and bifurcation theory. It leads to an accurate approximation to the analytical solution of the problem, showing the explicit interdependence of the various parameters. It offers considerable advantages over common numerical methods in that it enables us to gain insight into the system dynamics and to identify the important parameters that influence the overall motion.

In this paper, an extension of the methodology to the more complex problem of three-degree-of-freedom wing rock (i.e., roll, pitch, and yaw) is developed, leading to a deeper insight into the nonlinear dynamics of the aircraft.

II. Equations of Motion

The aircraft is assumed to be rigid, and its attitude deviations from the nominal values are considered to be small. During the motion of interest, the trajectory of the center of mass of the aircraft is assumed to be straight and horizontal and is not affected by its attitude motion.

The body-fixed ($X_bY_bZ_b$) and stability ($X_0Y_0Z_0$) axis systems are both used in this derivation. Both axis systems have their origin at the center of mass of the aircraft, and the X, Y, and Z axes follow the usual nose, right wing, and vertical orientations. The orientation of the stability axis system describes the nominal or unperturbed attitude of the aircraft, and, hence, in the nominal flight condition these two axis systems coincide with each other.

The angles describing the three consecutive rotations to bring the aircraft from its nominal position to its perturbed position are the Euler angles ψ, θ, and ϕ, that is, yaw, pitch, and roll angles, respectively. The angular rate of the aircraft in the body-fixed axes can be expressed in terms of the rate of change of the Euler angles by noting that

$$\boldsymbol{\omega} = p\boldsymbol{i}_{x_b} + q\boldsymbol{i}_{y_b} + r\boldsymbol{i}_{z_b}$$
$$= \dot{\psi}\boldsymbol{i}_{z_0} + \dot{\theta}\boldsymbol{i}_{y_1} + \dot{\phi}\boldsymbol{i}_{x_b} \tag{1}$$

The kinematic relations between $\{p, q, r\}$ and $\{\dot{\phi}, \dot{\theta}, \dot{\psi}\}$ for small angle and angular rate perturbations around the equilibrium position can then be approximated by

$$p \approx \dot{\phi} - \theta\dot{\psi}, \qquad q \approx \dot{\theta} + \phi\dot{\psi}, \qquad r \approx \dot{\psi} - \phi\dot{\theta} \tag{2}$$

Only aircraft with conventional configurations are considered in this analysis; therefore, the products of inertia $I_{xy} = I_{yz} = 0$. By using the Lagrangian approach and by defining the inertia ratios $n_1 \equiv I_{xz}/I_{xx}$, $n_2 \equiv I_{xz}/I_{yy}$, $n_3 \equiv I_{xz}/I_{zz}$, where I_{xx}, I_{yy}, and I_{zz} are the moments of inertia of the aircraft about X_b, Y_b, and Z_b, respectively, the following equations can be obtained:

$$\ddot{\phi} = \frac{1}{D}\Bigg[\left(1 + 2n_3\theta + 2\phi^2 + \frac{n_3}{n_1}\theta^2\right)\hat{f}_1 - \Bigg[\frac{n_1 n_3}{n_2}\phi + \left(n_1 n_3 + \frac{n_3}{n_2} - 1\right)\phi\theta\Bigg]\hat{f}_2 + \left(n_1 + \theta + n_1\phi^2\right)\hat{f}_3\Bigg]$$

$$\ddot{\theta} = \frac{1}{D}\Bigg[-\Bigg[n_3\phi + \left(n_2 n_3 + \frac{n_3}{n_1} - \frac{n_2}{n_1}\right)\phi\theta\Bigg]\hat{f}_1 + \left(1 - n_1 n_3 + \frac{n_3}{n_2}\phi^2\right)\hat{f}_2 - \left(n_1 n_2 + 1 - \frac{n_2}{n_3}\right)\phi\hat{f}_3\Bigg]$$

$$\ddot{\psi} = \frac{1}{D}\Bigg[\left(n_3 + \theta + n_3\phi^2\right)\hat{f}_1 - \left(n_1 n_3 + \frac{n_3}{n_2} - 1\right)\phi\hat{f}_2 + \Bigg[1 + \left(\frac{n_2}{n_3} - n_1 n_2\right)\phi^2\Bigg]\hat{f}_3\Bigg] \tag{3}$$

where

$$\frac{1}{D} = \frac{1}{(1 - n_1 n_3)(1 + \phi^2)^2} \approx \frac{1}{1 - n_1 n_3}(1 - 2\phi^2)$$

$$\hat{f}_1 \equiv \bar{Q}_1 + \left(1 + \frac{n_1}{n_2} + \frac{n_1}{n_3}\right)\dot{\theta}\dot{\psi} + \frac{n_1}{n_2}\phi\dot{\psi}^2 + \frac{n_1}{n_3}\phi\dot{\theta}^2 - n_1\theta\dot{\theta}\dot{\psi}$$

$$\hat{f}_2 \equiv \bar{Q}_2 - (\frac{n_2}{n_1} + 1 - \frac{n_2}{n_3})\dot{\phi}\dot{\psi} - n_2(\dot{\phi}^2 - \dot{\psi}^2) + \frac{n_2}{n_1}\theta\dot{\psi}^2 - 2\frac{n_2}{n_3}\phi\dot{\phi}\dot{\theta} + n_2\dot{\phi}\theta\dot{\psi}$$

$$\hat{f}_3 \equiv \bar{Q}_3 - \left(\frac{n_3}{n_1}\frac{n_3}{n_2} + 1\right)\dot{\phi}\dot{\theta} - 2n_3\dot{\theta}\dot{\psi} - n_3\phi\dot{\theta}^2 + n_3\dot{\phi}\theta\dot{\theta} - 2\frac{n_3}{n_2}\phi\dot{\phi}\dot{\psi} - 2\frac{n_3}{n_1}\theta\dot{\theta}\dot{\psi} \tag{4}$$

The Q_i in the equations are generalized forces contributed by the aerodynamics, which are derived next.

Following Ref. 14, the purpose of the aerodynamic derivation here is not to find the exact formulation or values of the coefficients, but rather to obtain approximate mathematical expressions, which are suitable for later analysis. The values of the coefficients in the expressions could be calculated from fitting the expressions obtained to the aerodynamic data from the experiments or from more sophisticated aerodynamic computations. For this reason, strip theory aerodynamics are utilized. Details can be found in Ref. 14, and for brevity, only some important points are mentioned here.

The local lift and drag coefficients $c_L(y)$ and $c_D(y)$ are assumed to have cubic functional relationship with the local effective angle of attack $\alpha_e(y)$ as follows:

$$c_L = c_{L_0} + c_{L_1}\alpha_e + c_{L_2}\alpha_e^2 + c_{L_3}\alpha_e^3$$
$$c_D = c_{D_0} + c_{D_1}\alpha_e + c_{D_2}\alpha_e^2 + c_{D_3}\alpha_e^3 \tag{5}$$

where in the preceding equations the dependence of the coefficients and α on the spanwise location y has been dropped. The effective angle-of-attack distribution is influenced by the nominal angle of attack α_0, p, q, r, angle of sideslip β, deviation from the nominal angle-of-attack α, and time-dependent effects $\dot{\alpha}$ and $\dot{\beta}$. Because only small deviations from the nominal condition are considered, the contributions of the just-mentioned factors on the effective angle-of-attack distribution can be expressed using the linear relation as follows:

$$\alpha_e(y) = \alpha_1(y) + \frac{\partial\alpha_e(y)}{\partial p}p + \frac{\partial\alpha_e(y)}{\partial\beta}\beta + \frac{\partial\alpha_e(y)}{\partial\dot{\beta}}\dot{\beta} + \frac{\partial\alpha_e(y)}{\partial\alpha}\alpha + \frac{\partial\alpha_e(y)}{\partial q}q + \frac{\partial\alpha_e(y)}{\partial\dot{\alpha}}\dot{\alpha} + \frac{\partial\alpha_e(y)}{\partial r}r \tag{6}$$

where $\alpha_1(y)$ indicates the contribution of the nominal angle of attack α_0. For simplicity, the spanwise angle-of-attack distribution as a result of p, r, β, and $\dot{\beta}$ is assumed to be antisymmetric and the one as a result of α_0, q, α, and $\dot{\alpha}$ is assumed to be symmetric.

By substituting Eq. (6) into Eq. (5) and then rearranging, we obtain lengthy equations expressing the lift and drag forces in terms of the variables $p, \beta, \dot{\beta}, \alpha, q, \dot{\alpha}$, and r. The Q_i can then be calculated from the variation of the work done by the aerodynamic forces caused by the variation in displacement, $Q_i = \delta W_i/\delta\gamma_i$.

Then, by the assumption that the aircraft trajectory is not influenced by the attitude motion,

$$\alpha \approx \theta, \qquad \dot{\alpha} \approx q$$
$$\dot{\beta} \approx p\sin\alpha_0 - r\cos\alpha_0 + (g/V)\phi\cos\alpha_0 \tag{7}$$

These approximations together with the kinematic relations (2) can be used to express the equations of motion only in three variables, that is, β, ϕ, and θ:

$$\begin{aligned}\ddot{\beta} + \omega_1^2\beta = {} & \tilde{\eta}_1\dot{\beta} + \tilde{\kappa}_2\phi + \tilde{\eta}_2\dot{\phi} + \tilde{e}_1\phi^3 + \tilde{e}_2\phi^2\dot{\phi} + \tilde{e}_3\phi\dot{\phi}^2 + \tilde{e}_4\dot{\phi}^3 \\ & + \tilde{e}_5\beta^3 + \tilde{e}_6\beta^2\dot{\beta} + \tilde{e}_7\beta\dot{\beta}^2 + \tilde{e}_8\dot{\beta}^3 + \tilde{e}_9\phi^2\beta + \tilde{e}_{10}\phi^2\dot{\beta} \\ & + \tilde{e}_{11}\dot{\phi}^2\beta + \tilde{e}_{12}\dot{\phi}^2\dot{\beta} + \tilde{e}_{13}\phi\beta^2 + \tilde{e}_{14}\phi\dot{\beta}^2 + \tilde{e}_{15}\dot{\phi}\beta^2 + \tilde{e}_{16}\dot{\phi}\dot{\beta}^2 \\ & + \tilde{e}_{17}\phi\theta + \tilde{e}_{18}\phi\dot{\theta} + \tilde{e}_{19}\dot{\phi}\theta + \tilde{e}_{20}\dot{\phi}\dot{\theta} + \tilde{e}_{21}\theta\beta + \tilde{e}_{22}\theta\dot{\beta} + \tilde{e}_{23}\dot{\theta}\beta \\ & + \tilde{e}_{24}\dot{\theta}\dot{\beta} + \tilde{e}_{25}\phi\theta^2 + \tilde{e}_{26}\phi\dot{\theta}^2 + \tilde{e}_{27}\dot{\phi}\theta^2 + \tilde{e}_{28}\dot{\phi}\dot{\theta}^2 + \tilde{e}_{29}\theta^2\beta \\ & + \tilde{e}_{30}\theta^2\dot{\beta} + \tilde{e}_{31}\dot{\theta}^2\beta + \tilde{e}_{32}\dot{\theta}^2\dot{\beta} + \tilde{e}_{33}\phi\dot{\phi}\beta + \tilde{e}_{34}\phi\dot{\phi}\dot{\beta} + \tilde{e}_{35}\phi\beta\dot{\beta} \\ & + \tilde{e}_{36}\dot{\phi}\beta\dot{\beta} + \tilde{e}_{37}\phi\theta\dot{\theta} + \tilde{e}_{38}\dot{\phi}\theta\dot{\theta} + \tilde{e}_{39}\theta\dot{\theta}\beta + \tilde{e}_{40}\theta\dot{\theta}\dot{\beta}\end{aligned}$$

$$\begin{aligned}\ddot{\phi} = {} & \tilde{\kappa}_1\beta + \tilde{\kappa}_3\phi + \tilde{\xi}_1\dot{\phi} + \tilde{\xi}_2\dot{\beta} + \tilde{c}_1\phi^3 + \tilde{c}_2\phi^2\dot{\phi} + \tilde{c}_3\phi\dot{\phi}^2 + \tilde{c}_4\dot{\phi}^3 \\ & + \tilde{c}_5\beta^3 + \tilde{c}_6\beta^2\dot{\beta} + \tilde{c}_7\beta\dot{\beta}^2 + \tilde{c}_8\dot{\beta}^3 + \tilde{c}_9\phi^2\beta + \tilde{c}_{10}\phi^2\dot{\beta} + \tilde{c}_{11}\dot{\phi}^2\beta \\ & + \tilde{c}_{12}\dot{\phi}^2\dot{\beta} + \tilde{c}_{13}\phi\beta^2 + \tilde{c}_{14}\phi\dot{\beta}^2 + \tilde{c}_{15}\dot{\phi}\beta^2 + \tilde{c}_{16}\dot{\phi}\dot{\beta}^2 + \tilde{c}_{17}\phi\theta \\ & + \tilde{c}_{18}\phi\dot{\theta} + \tilde{c}_{19}\dot{\phi}\theta + \tilde{c}_{20}\dot{\phi}\dot{\theta} + \tilde{c}_{21}\theta\beta + \tilde{c}_{22}\theta\dot{\beta} + \tilde{c}_{23}\dot{\theta}\beta + \tilde{c}_{24}\dot{\theta}\dot{\beta} \\ & + \tilde{c}_{25}\phi\theta^2 + \tilde{c}_{26}\phi\dot{\theta}^2 + \tilde{c}_{27}\dot{\phi}\theta^2 + \tilde{c}_{28}\dot{\phi}\dot{\theta}^2 + \tilde{c}_{29}\theta^2\beta \\ & + \tilde{c}_{30}\theta^2\dot{\beta} + \tilde{c}_{31}\dot{\theta}^2\beta + \tilde{c}_{32}\dot{\theta}^2\dot{\beta} + \tilde{c}_{33}\phi\dot{\phi}\beta + \tilde{c}_{34}\phi\dot{\phi}\dot{\beta} + \tilde{c}_{35}\phi\beta\dot{\beta} \\ & + \tilde{c}_{36}\dot{\phi}\beta\dot{\beta} + \tilde{c}_{37}\phi\theta\dot{\theta} + \tilde{c}_{38}\dot{\phi}\theta\dot{\theta} + \tilde{c}_{39}\theta\dot{\theta}\beta + \tilde{c}_{40}\theta\dot{\theta}\dot{\beta}\end{aligned}$$

$$
\begin{aligned}
\ddot{\theta} + \Omega^2\theta = {} & \tilde{\nu}\dot{\theta} + \tilde{d}_4\phi^2 + \tilde{d}_5\phi\dot{\phi} + \tilde{d}_6\dot{\phi}^2 + \tilde{d}_7\beta^2 + \tilde{d}_8\beta\dot{\beta} + \tilde{d}_9\dot{\beta}^2 \\
& + \tilde{d}_{10}\phi\beta + \tilde{d}_{11}\phi\dot{\beta} + \tilde{d}_{12}\dot{\phi}\beta + \tilde{d}_{13}\dot{\phi}\dot{\beta} + \tilde{d}_1\theta^2 + \tilde{d}_2\theta\dot{\theta} + \tilde{d}_3\dot{\theta}^2 \\
& + \tilde{d}_{14}\theta^3 + \tilde{d}_{15}\theta^2\dot{\theta} + \tilde{d}_{16}\theta\dot{\theta}^2 + \tilde{d}_{17}\dot{\theta}^3 + \tilde{d}_{18}\phi^2\theta + \tilde{d}_{19}\phi^2\dot{\theta} \\
& + \tilde{d}_{20}\dot{\phi}^2\theta + \tilde{d}_{21}\dot{\phi}^2\dot{\theta} + \tilde{d}_{22}\theta\beta^2 + \tilde{d}_{23}\theta\dot{\beta}^2 + \tilde{d}_{24}\dot{\theta}\beta^2 + \tilde{d}_{25}\dot{\theta}\dot{\beta}^2 \\
& + \tilde{d}_{26}\phi\dot{\phi}\theta + \tilde{d}_{27}\phi\dot{\phi}\dot{\theta} + \tilde{d}_{28}\theta\beta\dot{\beta} + \tilde{d}_{29}\dot{\theta}\beta\dot{\beta} + \tilde{d}_{30}\phi\theta\beta + \tilde{d}_{31}\phi\theta\dot{\beta} \\
& + \tilde{d}_{32}\phi\dot{\theta}\beta + \tilde{d}_{33}\phi\dot{\theta}\dot{\beta} + \tilde{d}_{34}\dot{\phi}\theta\beta + \tilde{d}_{35}\dot{\phi}\theta\dot{\beta} + \tilde{d}_{36}\dot{\phi}\dot{\theta}\beta + \tilde{d}_{37}\dot{\phi}\dot{\theta}\dot{\beta}
\end{aligned}
\tag{8}
$$

The interested reader can consult Ref. 17 for further details.

III. Dynamics Analysis

The analytical methodology developed in Refs. 13, 14, and 17, which utilizes the MTS method in conjunction with center manifold reduction techniques and the bifurcation theory, is used to solve the problem. In this approach, the MTS method is used to reduce the complicated equations of motion to a form where center manifold reduction techniques and bifurcation theory can be readily applied. This technique allows us to analyze the properties of the system and obtain the approximation of the solution in a parametric form.

A. MTS Analysis

The MTS technique invoked here is based on the development of Ramnath.[15,16] The fundamental concept is is to extend the time as the independent variable into several time scales and then solve the resulting equations in each time scale systematically. The concept of the extension is illustrated in Fig. 1. For the sake of brevity, the details of the technique will not be discussed here. The interested reader can consult the references for detail.

To facilitate the MTS analysis, the equations of motion (8) are first parameterized. First, because only small motions around the equilibrium conditions are considered, then for each equation

$$
\lim_{x \to \bar{0}} \frac{|N(\boldsymbol{x})|}{|\boldsymbol{x}|} \to 0 \tag{9}
$$

where $\boldsymbol{x} = \{\beta \;\; \phi \;\; \theta \;\; \dot{\beta} \;\; \dot{\phi} \;\; \dot{\theta}\}^T$ and $N(\boldsymbol{x})$ contains all the nonlinear terms in the equation. This is equivalent to saying that $N(\boldsymbol{x}) = \boldsymbol{o}(\boldsymbol{x})$. Second, in the vicinity of the wing rock onset, the magnitudes of the lateral motion are much larger than those of the longitudinal motion. This means that mathematically $\mathcal{O}(\theta) = \epsilon\mathcal{O}(\phi)$ and $\mathcal{O}(\theta) = \epsilon\mathcal{O}(\beta)$, where $0 < \epsilon \ll 1$. Third, in near wing-rock case the damping terms are usually small. Finally, it is assumed that $g/\omega V$ is small $[\mathcal{O}(\epsilon)]$, where ω denotes the dominant frequency of the rotational motion. This ratio can be interpreted as the ratio of the time scale of the translational motion to that of the rotational motion. This assumption implies that the rotational motion time scale is much faster than the translational time scale. Thus, the equations of motion can be written as as follows:

$$
\begin{aligned}
& \ddot{\beta} + \omega_1^2\beta = \epsilon[\eta_1\dot{\beta} + \kappa_2\phi + \eta_2\dot{\phi} + f_1(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta})] \\
& \ddot{\phi} = \kappa_1\beta + \epsilon[\kappa_3\phi + \xi_1\dot{\phi} + \xi_2\dot{\beta} + f_2(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta})] \\
& \ddot{\theta} + \Omega^2\theta = g(\beta, \dot{\beta}, \phi, \dot{\phi}) + \epsilon[\nu\dot{\theta} + f_3(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta})]
\end{aligned}
\tag{10}
$$

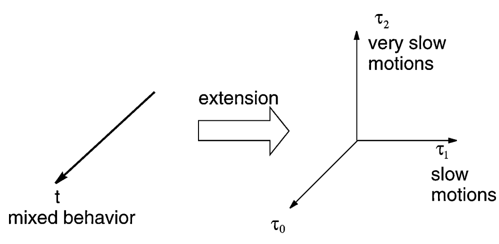

Fig. 1 Concept of extension.

where

$$
\begin{aligned}
g(\beta, \dot{\beta}, \phi, \dot{\phi}) = {} & d_4\phi^2 + d_5\phi\dot{\phi} + d_6\dot{\phi}^2 + d_7\beta^2 + d_8\beta\dot{\beta} + d_9\dot{\beta}^2 \\
& + d_{10}\phi\beta + d_{11}\phi\dot{\beta} + d_{12}\dot{\phi}\beta + d_{13}\dot{\phi}\dot{\beta} \\
f_1(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta}) = {} & e_1\phi^3 + e_2\phi^2\dot{\phi} + e_3\phi\dot{\phi}^2 + e_4\dot{\phi}^3 + e_5\beta^3 \\
& + e_6\beta^2\dot{\beta} + e_7\beta\dot{\beta}^2 + e_8\dot{\beta}^3 + e_9\phi^2\beta + e_{10}\phi^2\dot{\beta} + e_{11}\dot{\phi}^2\beta \\
& + e_{12}\dot{\phi}^2\dot{\beta} + e_{13}\phi\beta^2 + e_{14}\phi\dot{\beta}^2 + e_{15}\dot{\phi}\beta^2 + e_{16}\dot{\phi}\dot{\beta}^2 + e_{17}\phi\theta \\
& + e_{18}\phi\dot{\theta} + e_{19}\dot{\phi}\theta + e_{20}\dot{\phi}\dot{\theta} + e_{21}\theta\beta + e_{22}\theta\dot{\beta} + e_{23}\dot{\theta}\beta + e_{24}\dot{\theta}\dot{\beta} \\
& + e_{25}\phi\theta^2 + e_{26}\phi\dot{\theta}^2 + e_{27}\dot{\phi}\theta^2 + e_{28}\dot{\phi}\dot{\theta}^2 + e_{29}\theta^2\beta + e_{30}\theta^2\dot{\beta} \\
& + e_{31}\dot{\theta}^2\beta + e_{32}\dot{\theta}^2\dot{\beta} + e_{33}\phi\dot{\phi}\beta + e_{34}\phi\dot{\phi}\dot{\beta} + e_{35}\phi\beta\dot{\beta} + e_{36}\dot{\phi}\beta\dot{\beta} \\
& + e_{37}\phi\theta\dot{\theta} + e_{38}\dot{\phi}\theta\dot{\theta} + e_{39}\theta\dot{\theta}\beta + e_{40}\theta\dot{\theta}\dot{\beta} \\
f_2(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta}) = {} & c_1\phi^3 + c_2\phi^2\dot{\phi} + c_3\phi\dot{\phi}^2 + c_4\dot{\phi}^3 + c_5\beta^3 \\
& + c_6\beta^2\dot{\beta} + c_7\beta\dot{\beta}^2 + c_8\dot{\beta}^3 + c_9\phi^2\beta + c_{10}\phi^2\dot{\beta} + c_{11}\dot{\phi}^2\beta \\
& + c_{12}\dot{\phi}^2\dot{\beta} + c_{13}\phi\beta^2 + c_{14}\phi\dot{\beta}^2 + c_{15}\dot{\phi}\beta^2 + c_{16}\dot{\phi}\dot{\beta}^2 + c_{17}\phi\theta \\
& + c_{18}\phi\dot{\theta} + c_{19}\dot{\phi}\theta + c_{20}\dot{\phi}\dot{\theta} + c_{21}\theta\beta + c_{22}\theta\dot{\beta} + c_{23}\dot{\theta}\beta + c_{24}\dot{\theta}\dot{\beta} \\
& + c_{25}\phi\theta^2 + c_{26}\phi\dot{\theta}^2 + c_{27}\dot{\phi}\theta^2 + c_{28}\dot{\phi}\dot{\theta}^2 + c_{29}\theta^2\beta + c_{30}\theta^2\dot{\beta} \\
& + c_{31}\dot{\theta}^2\beta + c_{32}\dot{\theta}^2\dot{\beta} + c_{33}\phi\dot{\phi}\beta + c_{34}\phi\dot{\phi}\dot{\beta} + c_{35}\phi\beta\dot{\beta} + c_{36}\dot{\phi}\beta\dot{\beta} \\
& + c_{37}\phi\theta\dot{\theta} + c_{38}\dot{\phi}\theta\dot{\theta} + c_{39}\theta\dot{\theta}\beta + c_{40}\theta\dot{\theta}\dot{\beta} \\
f_3(\beta, \dot{\beta}, \phi, \dot{\phi}, \theta, \dot{\theta}) = {} & d_1\theta^2 + d_2\theta\dot{\theta} + d_3\dot{\theta}^2 + d_{14}\theta^3 + d_{15}\theta^2\dot{\theta} \\
& + d_{16}\theta\dot{\theta}^2 + d_{17}\dot{\theta}^3 + d_{18}\phi^2\theta + d_{19}\phi^2\dot{\theta} + d_{20}\dot{\phi}^2\theta + d_{21}\dot{\phi}^2\dot{\theta} \\
& + d_{22}\theta\beta^2 + d_{23}\theta\dot{\beta}^2 + d_{24}\dot{\theta}\beta^2 + d_{25}\dot{\theta}\dot{\beta}^2 + d_{26}\phi\dot{\phi}\theta + d_{27}\phi\dot{\phi}\dot{\theta} \\
& + d_{28}\theta\beta\dot{\beta} + d_{29}\dot{\theta}\beta\dot{\beta} + d_{30}\phi\theta\beta + d_{31}\phi\theta\dot{\beta} + d_{32}\phi\dot{\theta}\beta + d_{33}\phi\dot{\theta}\dot{\beta} \\
& + d_{34}\dot{\phi}\theta\beta + d_{35}\dot{\phi}\theta\dot{\beta} + d_{36}\dot{\phi}\dot{\theta}\beta + d_{37}\dot{\phi}\dot{\theta}\dot{\beta}
\end{aligned}
\tag{11}
$$

The relations between the coefficients in Eqs. (10) and (8) can be found by the one-to-one correspondence between their terms.[17]

The proper extension of time can be determined using the principle of minimal simplification,[18] which in this case leads to

$$
t \to \{\tau_0, \tau_1, \tau_2\}, \qquad \tau_0 = t, \qquad \tau_1 = \epsilon^{\frac{1}{2}}t, \qquad \tau_2 = \epsilon t \tag{12}
$$

The dependent variables are then extended as

$$
\begin{aligned}
\beta(t) &\to \beta_0(\tau_0, \tau_1, \tau_2) + \epsilon\beta_1(\tau_0, \tau_1, \tau_2) + \cdots \\
\phi(t) &\to \phi_0(\tau_0, \tau_1, \tau_2) + \epsilon\phi_1(\tau_0, \tau_1, \tau_2) + \cdots \\
\theta(t) &\to \theta_0(\tau_0, \tau_1, \tau_2) + \epsilon\theta_1(\tau_0, \tau_1, \tau_2) + \cdots
\end{aligned}
\tag{13}
$$

Substitution into Eq. (10) leads to the extended perturbation equation of $\mathcal{O}(1)$:

$$
\begin{gathered}
\frac{\partial^2\beta_0}{\partial\tau_0^2} + \omega_1^2\beta_0 = 0, \qquad \frac{\partial^2\phi_0}{\partial\tau_0^2} = \kappa_1\beta_0 \\
\frac{\partial^2\theta_0}{\partial\tau_0^2} + \Omega^2\theta_0 = g(\beta_0, \dot{\beta}_0, \phi_0, \dot{\phi}_0)
\end{gathered}
\tag{14}
$$

The solution of the first equation in Eq. (14) is

$$
\beta_0 = A_1(\tau_1, \tau_2)\sin\Psi_1, \qquad \Psi_1 \equiv \omega_1\tau_0 + B_1(\tau_1) \tag{15}
$$

The substitution of this solution into the second equation in Eq. (14) yields

$$
\phi_0 = \phi_{0_1} + \phi_{0_2} \tag{16}
$$

where

$$\phi_{0_1} = -(\kappa_1/\omega_1^2)A_1(\tau_1,\tau_2)\sin\Psi_1, \qquad \phi_{0_2} = C(\tau_1,\tau_2) \tag{17}$$

Here ϕ_{0_1} and ϕ_{0_2} are treated as two independent solutions. Although we are dealing with a nonlinear system, this treatment is justifiable because these solutions are obtained from a set of linear partial differential equations.

The substitution of β_0 and ϕ_{0_1} into the third equation in Eq. (14) yields

$$\theta_0 = \theta_{0_1} + \theta_{0_2} \tag{18}$$

where

$$\theta_{0_1} = A_2(\tau_1,\tau_2)\sin\Psi_2, \qquad \Psi_2 \equiv \Omega\tau_0 + B_2(\tau_1,\tau_2)$$

$$\theta_{0_2} = m_0A_1^2(\tau_1,\tau_2) + m_1A_1^2(\tau_1,\tau_2)\cos 2\Psi_1 + m_2A_1^2(\tau_1,\tau_2)\sin 2\Psi_1 \tag{19}$$

with

$$m_0 = (1/2\Omega^2)\left[(\kappa_1/\omega_1^2)^2(d_1 + d_3\omega_1^2) + (d_4 + d_6\omega_1^2) - (\kappa_1/\omega_1^2)(d_7 + d_{10}\omega_1^2)\right]$$

$$m_1 = \left[1/2(\Omega^2-\omega_1^2)\right]\left[-(\kappa_1/\omega_1^2)^2(d_1 - d_3\omega_1^2) - (d_4 - d_6\omega_1^2) + (\kappa_1/\omega_1^2)(d_7 - d_{10}\omega_1^2)\right]$$

$$m_2 = \left[1/2(\Omega^2-\omega_1^2)\right]\left[(\kappa_1/\omega_1^2)^2 d_2\omega_1 + d_5\omega_1 - (\kappa_1/\omega_1^2)(d_8 + d_9)\right] \tag{20}$$

Analysis of the order $\mathcal{O}(\epsilon^{1/2})$ leads us to the finding that A_1, B_1, A_2, and B_2 are functions of τ_2 only (not functions of τ_1).

Order ϵ equation for β is

$$\mathcal{O}(\epsilon)\colon \frac{\partial^2\beta_1}{\partial\tau_0^2} + \omega_1^2\beta_1 = -2\frac{\partial^2\beta_0}{\partial\tau_0\partial\tau_2} - \frac{\partial^2\beta_0}{\partial\tau_1^2} + \eta_1\frac{\partial\beta_0}{\partial\tau_0} + \kappa_2\phi_0 + \eta_1\frac{\partial\phi_0}{\partial\tau_0} + f_{1_0} = [Coef_1]\cos\Psi_1 + [Coef_2]\sin\Psi_1 + \cdots \tag{21}$$

where f_{1_0} is the extended version of the nonlinear function f_1 of $\mathcal{O}(1)$, where the fourth- and higher-order terms are neglected. The secular terms in the β_1 solution caused by $\cos\Psi_1$ and $\sin\Psi_1$ terms are eliminated by the uniformity condition.

$$\frac{dA_1}{d\tau_2} = \frac{\mu}{2}A_1 + p_1A_1^3 + p_2A_1A_2^2$$

$$\frac{dB_1}{d\tau_1} = p_3 + p_4A_1^2 + p_5A_2^2 \tag{22}$$

where

$$\mu = \frac{1}{2}\left(\eta_1 - \frac{\kappa_1}{\omega_1^2}\eta_2\right)$$

$$p_1 = -\frac{1}{8}\frac{\kappa_1^3}{\omega_1^6}e_2 - \frac{3}{8}\frac{\kappa_1^3}{\omega_1^4}e_4 + \frac{1}{8}e_6 + \frac{3}{8}\omega_1^2e_8 + \frac{1}{8}\frac{\kappa_1^2}{\omega_1^4}e_{10} + \frac{3}{8}\frac{\kappa_1^2}{\omega_1^2}e_{12} - \frac{1}{8}\frac{\kappa_1}{\omega_1^2}e_{15} - \frac{3}{8}\kappa_1e_{16} - \frac{1}{4}\frac{\kappa_1m_3}{\omega_1^3(\Omega^2-4\omega_1^2)}e_{17} + \frac{1}{2}\frac{\kappa_1m_2}{\omega_1^2(\Omega^2-4\omega_1^2)}e_{18} - \frac{1}{2}\left[\frac{\kappa_1m_1}{\omega_1^2\Omega^2} + \frac{1}{2}\frac{\kappa_1m_2}{\omega_1^2(\Omega^2-4\omega_1^2)}\right]e_{19} - \frac{1}{2}\frac{\kappa_1m_3}{\omega_1(\Omega^2-4\omega_1^2)}e_{20} + \frac{1}{4}\frac{m_3}{\omega_1(\Omega^2-4\omega_1^2)}e_{21} + \frac{1}{2}\left(\frac{m_1}{\Omega^2} + \frac{m_2}{\Omega^2-4\omega_1^2}\right)e_{22} - \frac{1}{2}\frac{m_2}{(\Omega^2-4\omega_1^2)}e_{23} + \frac{1}{2}\frac{m_3\omega_2}{(\Omega^2-4\omega_1^2)}e_{24} + \frac{1}{8}\frac{\kappa_1^2}{\omega_1^4}e_{33} - \frac{1}{8}\frac{\kappa_1}{\omega_1^2}e_{35}$$

$$p_2 = -\frac{1}{4}\frac{\kappa_1}{\omega_1^2}e_{27} - \frac{1}{4}\frac{\kappa_1\Omega^2}{\omega_1^2}e_{28} + \frac{1}{4}e_{30} + \frac{1}{4}\Omega^2e_{32}$$

$$p_3 = -\frac{1}{4}\frac{\kappa_1}{\omega_1^2}e_2 - \frac{3}{4}\frac{\kappa_1\omega_1^2}{\omega_1^2}e_4 + \frac{1}{4}e_{10} + \frac{1}{4}\omega_1^2e_{12}$$

$$p_4 = \frac{\kappa_1\kappa_2}{2\omega_1^3}$$

$$p_5 = \frac{3}{8}\frac{\kappa_1^3}{\omega_1^7}e_1 + \frac{1}{8}\frac{\kappa_1^3}{\omega_1^5}e_3 - \frac{3}{8}\frac{1}{\omega_2}e_5 - \frac{1}{4}\omega_2e_7 - \frac{3}{8}\frac{\kappa_1^2}{\omega_1^5}e_9 - \frac{1}{8}\frac{\kappa_1^2}{\omega_1^3}e_{11} + \frac{3}{8}\frac{\kappa_1}{\omega_1^3}e_{13} + \frac{1}{8}\frac{\kappa_1}{\omega_1}e_{14} + \frac{1}{2}\left[\frac{\kappa_1m_1}{\omega_1^3\Omega^2} - \frac{1}{2}\frac{\kappa_1m_2}{\omega_1^3(\Omega^2-4\omega_1^2)}\right]e_{17} - -\frac{1}{2}\left[\frac{m_1}{\omega_1\Omega^2} - \frac{m_2}{\omega_1(\Omega^2-4\omega_1^2)}\right]e_{21} - \frac{1}{2}\frac{\kappa_1m_3}{\omega_1^2(\Omega^2-4\omega_1^2)}e_{18} + \frac{1}{4}\frac{\kappa_1m_3}{\omega_1^2(\Omega^2-4\omega_1^2)}e_{19} - \frac{1}{2}\frac{\kappa_1m_2}{\omega_1(\Omega^2-4\omega_1^2)}e_{20} - \frac{1}{4}\frac{m_3}{\Omega^2-4\omega_1^2}e_{22} + \frac{1}{2}\frac{m_3}{\Omega^2-4\omega_1^2}e_{23} + \frac{1}{2}\frac{\omega_1m_2}{\Omega^2-4\omega_1^2}e_{24} - \frac{1}{8}\frac{\kappa_1^2}{\omega_1^3}e_{34} + \frac{1}{8}\frac{\kappa_1}{\omega_2}e_{36} \tag{23}$$

The first equation in Eq. (22) is the lateral amplitude equation while the second one gives the phase correction of this lateral oscillation. These equations describe the slowly changing behavior of the amplitude and phase of this particular mode. Notice that these differential equations depend on the longitudinal mode through A_2.

The $\mathcal{O}(\epsilon)$ terms in the longitudinal equation lead to the expression

$$\mathcal{O}(\epsilon)\colon \frac{\partial^2\theta_1}{\partial\tau_0^2} + \Omega_1^2\theta_1 = -2\frac{\partial^2\theta_0}{\partial\tau_0\partial\tau_2} + \nu\frac{\partial\theta_0}{\partial\tau_0} + g_2 + f_{3_0} = [Coef_3]\cos\Psi_2 + [Coef_4]\sin\Psi_2 + \cdots \tag{24}$$

Again, the secular terms caused by $\cos\Psi_2$ and $\sin\Psi_2$ are eliminated by requiring the coefficients of these terms to be zero. Therefore,

$$\frac{dA_2}{d\tau_2} = \frac{1}{2}\nu A_2 + q_1A_2^3 + q_2A_1^2A_2, \qquad \frac{dB_2}{d\tau_2} = q_3A_1^2 + q_4A_2^2 \tag{25}$$

where

$$q_1 = \frac{1}{8}d_{15} + \frac{3}{8}\Omega^2d_{17}$$

$$q_2 = \frac{1}{4}\frac{\kappa_1\Omega}{\omega_1^4}d_{19} + \frac{1}{4}\frac{\kappa_1^2}{\omega_1^2}d_{21} + \frac{1}{4}d_{24} + \frac{1}{4}\omega_1^2d_{25} - \frac{1}{4}\frac{\kappa_1}{\omega_1^2}d_{32} - \frac{1}{4}\kappa_1d_{37} + \frac{1}{2}\frac{m_1}{\Omega^2}$$

$$q_3 = -\frac{m_1}{\Omega^3}d_1 - \frac{1}{4}\frac{\kappa_1^2}{\Omega\omega_1^4}d_{18} - \frac{1}{4}\frac{1}{\Omega}d_{22} + \frac{1}{4}\frac{\kappa_1}{\Omega\omega_1^2}d_{30} - \frac{1}{4}\frac{\kappa_1^2}{\Omega\omega_1^2}d_{20} - \frac{1}{4}\frac{\omega_1^2}{\Omega}d_{23} + \frac{1}{4}\frac{\kappa_1}{\Omega}d_{35}$$

$$q_4 = -\frac{3}{8}\frac{1}{\Omega}d_{14} - \frac{1}{8}\Omega d_{16} \tag{26}$$

Equation (25) gives the amplitude and phase correction of the homogeneous longitudinal mode. This set together with Eq. (22) needs to be solved to get the amplitude and phase correction for the dominant lateral and longitudinal modes.

The other roll mode ϕ_{0_2} is obtained from the $\mathcal{O}(\epsilon)$ ϕ equation as

$$\frac{\partial^2 C}{\partial \tau_1^2} + \omega_2^2 C + u_1 C^3 = 0 \tag{27}$$

where

$$\omega_2^2 = \kappa_3 + \kappa_1\kappa_2\big/\omega_1^2$$

$$u_1 = c_1 + c_{17}\left(d_1\big/\Omega^2\right) + \left(\kappa_1\big/\omega_1^2\right)\left[e_1 + e_{17}\left(d_1\big/\Omega^2\right)\right] \tag{28}$$

Noting that the amplitude varies with the slower time scale τ_2 and $\partial C/\partial \tau_1 = 0$, Eq. (27) can be integrated to yield

$$\left(\frac{\partial C}{\partial \tau_1}\right)^2 = \frac{u_1}{2}\left[A_3^2(\tau_2) - C^2\right]\left[\frac{2\omega_2^2}{u_1} + A_3^2(\tau_2) + C^2\right] \tag{29}$$

where $A_3(\tau_2)$ is the amplitude of the motion. By separating the variables and then integrating, the solution of the preceding equation can be expressed in terms of the elliptic function cn as follows[19]:

$$\tau_1 = \frac{1}{\omega_2\sqrt{1 + \left(u_1\big/\omega_1^2\right)A_3^2(\tau_2)}}cn^{-1}\left[\frac{C}{A_3(\tau_2)}, k\right] \tag{30}$$

where k is the modulus, which in this case is given by

$$k = \frac{A_3^2(\tau_2)}{2\omega_2^2\big/u_1 + 2A_3^2(\tau_2)} \tag{31}$$

The inversion of Eq. (30) yields

$$C(\tau_1, \tau_2) = A_3(\tau_2)cn\left[\omega_2\sqrt{1 + \left(u_1\big/\omega_1^2\right)A_3^2(\tau_2)}\tau_1\right] \tag{32}$$

The period of the oscillation is given by[20]

$$P = \frac{4\chi}{\omega_2\sqrt{1 + \left(u_1\big/\omega_1^2\right)A_3^2}} \tag{33}$$

where

$$\chi = \tfrac{1}{2}\pi\left[1 + \left(\tfrac{1}{2}\right)^2 k^2 + (1\,.\,3/2\,.\,4)^2 k^4 + \cdots\right] \tag{34}$$

For small-amplitude motions, the modulus k is small, and the angular frequency can be written as

$$\omega_3 = 2\pi/P \approx \omega_2\left[1 + \tfrac{3}{8}\left(u_1\big/\omega_2^2\right)A_3^2(\tau_2)\right] \tag{35}$$

Also, for small k we can approximate the elliptic function (32) using the sinusoidal function with the same frequency as follows:

$$C(\tau_1, \tau_2) = A_3(\tau_2)\sin\left\{\omega_2\left[1 + \tfrac{3}{8}\left(u_1\big/\omega_2^2\right)A_3^2(\tau_2)\right]\tau_1 + B_3(\tau_2)\right\} \tag{36}$$

where A_3 and B_3 represent respectively the amplitude and phase corrections of the solution, which can be found from the terms of next order in the expansion of Eq. (10):

$$\frac{\mathrm{d}A_3}{\mathrm{d}\tau_2} \approx \frac{1}{2}\left(\xi_2 + \frac{\kappa_1\eta_2}{\omega_1^2}\right)A_3 + \frac{1}{8}c_2A_3^3, \qquad \frac{\mathrm{d}B_3}{\mathrm{d}\tau_2} = 0 \tag{37}$$

by neglecting the amplitude-dependent part of the frequency because we only deal with small-amplitude motions.

B. Bifurcation Analysis and Center Manifold Reduction

Consider first the A_3 equation given in Eq. (37), which is uncoupled from the other amplitude equations. For conventional aircraft, the coefficient c_2 can be approximated very well using

$$c_2 = -(n_1n_3/n_2)d_2\tan\alpha_0 \tag{38}$$

Note that in the preceding equation n_i are always positive. For angle of attack between 0 and 90 deg, $\tan\alpha_0$ is positive and d_2 is the dominant contribution to the pitch damping parameter. Only the case where $\nu < 0$ is considered here, where $\nu = \bar{q}c/I_{yy}(d_2 + d_3)$ (from the definition of ν). The coefficients d_2 and d_3 depend on the stability derivatives C_{m_q} and $C_{m_{\dot\alpha}}$, respectively. The contribution of $C_{m_{\dot\alpha}}$ to ν is much smaller than C_{m_q} in most situations. Therefore, the sign of ν is mostly determined by d_2. The assumption that $\nu < 0$ almost always implies that $d_2 < 0$, which is assumed to hold for the rest of the discussion.

The equilibria for the A_3 equation consists of $A_3 = 0$ and $A_3 = \sqrt{(-4\vartheta/c_2)}$, where $\vartheta = (\xi_2 + \kappa_1\eta_2/\omega_1^2)$. Figure 2 shows the equilibria as the ϑ axis and the parabola $\vartheta = \frac{1}{4}c_2A_3^2$. The equilibrium $A_3 = 0$ is stable if $\vartheta < 0$ and unstable if $\vartheta > 0$, whereas the equilibria $\vartheta = -\frac{1}{4}c_2A_3^2$ are stable for $\vartheta > 0$ and unstable for $\vartheta < 0$. The only stable branch of equilibria is the the one on the negative ϑ axis (subcritical Hopf bifurcation). The nominal condition of the aircraft is stable when $\vartheta < 0$. The aircraft motion diverges when $\vartheta > 0$.

Consider now the A_1 and A_2 equations. They are coupled, and the center manifold reduction technique[21,22] is used to approach the problem. This technique is based on the idea that the dynamics of the system are represented asymptotically by the dynamics on the center manifold, generally of lower dimension than the complete system equations.

The center manifold of an equilibrium point is an invariant manifold that contains the equilibrium point and is tangent to the center eigenspace of the linearized system. In this case, the amplitude equations are of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}), \qquad \dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y} + \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y}) \tag{39}$$

where $\boldsymbol{x} \in \boldsymbol{R}^l$, $\boldsymbol{y} \in \boldsymbol{R}^m$, $\boldsymbol{A}$, and $\boldsymbol{B}$ are constant matrices such that $\Re(\lambda_i[\boldsymbol{A}]) = 0$; $i = 1, \ldots, l$ and $\Re(\lambda_i[\boldsymbol{B}]) < 0$; $i = 1, \ldots, m$. The linearized equation around the origin has two obvious eigenspaces, namely, $\boldsymbol{x} = 0$ and $\boldsymbol{y} = 0$, which represent stable and center eigenspaces, respectively. If the functions $\boldsymbol{p}$ and $\boldsymbol{q}$ along with their Jacobians vanish at the origin and a local center manifold $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$ for $|\boldsymbol{x}| < \delta$, $0 < \delta \ll 1$ can be found, where $\boldsymbol{h}(0) = \nabla_h(0) = 0$ (Refs. 21 and 22), then the flow on the center manifold is governed by the l-dimensional system

$$\dot{z} = \boldsymbol{A}z + \boldsymbol{p}[z, \boldsymbol{h}(z)] \tag{40}$$

Equation (40), which is an l-dimensional equation, contains all of the necessary information to determine the asymptotic behavior of the solutions of Eq. (39), which is an $(l + m)$-dimensional system.[22]

If we consider the case where μ is small but not zero and treat μ as a trivial dependent variable, then the coupled A_1 and A_2 equations

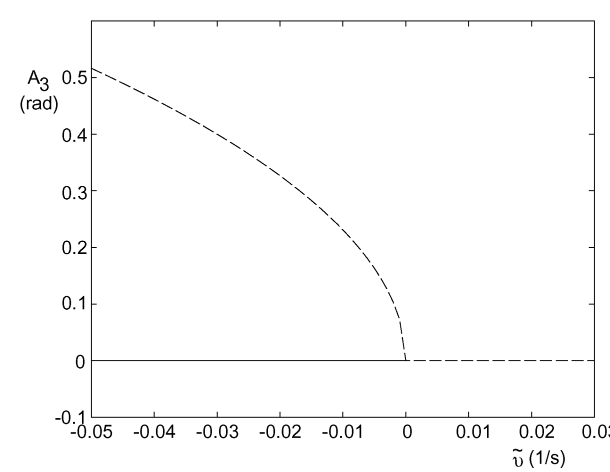

Fig. 2 Bifurcation diagram for A_3 equation.

in Eqs. (22) and (25) become

$$\frac{dA_1}{d\tau_2} = \frac{1}{2}\mu A_1 + p_1 A_1^3 + p_2 A_1 A_2^2$$

$$\frac{dA_2}{d\tau_2} = \frac{1}{2}\nu A_2 + q_1 A_2^3 + q_2 A_1^2 A_2, \qquad \frac{d\mu}{d\tau_2} = 0 \tag{41}$$

Note that in this formulation, the term $\frac{1}{2}\mu A_1$ is considered to be nonlinear. The equilibrium point of interest is the origin $(A_1, A_2, \mu) = (0, 0, 0)$. The linearization of the system (41) around the origin results in

$$\frac{dA_1}{d\tau_2} = 0, \qquad \frac{dA_2}{d\tau_2} = \frac{1}{2}\nu A_2, \qquad \frac{d\mu}{d\tau_2} = 0 \tag{42}$$

The eigenvalues of this linearized system are 0, ν, and 0. By the assumption $\nu < 0$, the A_2 axis is a stable manifold. We will find a center manifold

$$A_2 = h(A_1, \mu) \tag{43}$$

which satisfies $h(0,0) = (dh/dA_1)(0,0) = (dh/d\mu)(0,0) = 0$. To satisfy this requirement, $h = \mathcal{O}[(|A_1| + |\mu|)^n]$; $n > 1$. By differentiating Eq. (43) with respect to τ_1, we get

$$\frac{dA_2}{d\tau_1} = \frac{dh}{dA_1}\frac{dA_1}{d\tau_1} + \frac{dh}{d\mu}\frac{d\mu}{d\tau_1} \tag{44}$$

Then, by substituting $dA_1/d\tau_1$, $dA_2/d\tau_1$, and $d\mu/d\tau_1$ from Eq. (41) into the preceding equation, we obtain

$$\frac{dh}{dA_1} = \frac{\frac{1}{2}\nu h + q_1 h^3 + q_2 A_1^2 h}{\frac{1}{2}\mu A_1 + p_1 A_1^3 + p_2 A_1 h^2} \tag{45}$$

Because solving this equation is very difficult, we simplify it by remembering that $h = \mathcal{O}[(|A_1| + |\mu|)^n]$, $n > 1$ and neglecting terms in the numerator and denominator of $\mathcal{O}[(|A_1| + |\mu|)^k]$, $k > 3$. Doing so, we get

$$\frac{dh}{dA_1} \approx \frac{\frac{1}{2}\nu h}{\frac{1}{2}\mu A_1 + p_1 A_1^3} \tag{46}$$

It can be shown that the solution of the simplified equation is

$$h(A_1, \mu) = C\left[\frac{A_1^2}{\left(\frac{1}{2}\mu + p_1 A_1^2\right)}\right]^{\frac{1}{2}(\nu/\mu)} \tag{47}$$

where C is a constant to be determined from the condition $h(0,0) = (dh/dA_1)(0,0) = (dh/d\mu)(0,0) = 0$. This condition can only be satisfied when $C = 0$. Therefore, the center manifold of the system is $A_2 = 0$, which is the $A_1 - \mu$ plane.

The reduced system is then given by

$$\frac{dA_1}{d\tau_2} = \frac{1}{2}\mu A_1 + p_1 A_1^3, \qquad \frac{d\mu}{d\tau_2} = 0 \tag{48}$$

The equilibria of this system consist of the μ axis and the parabola $\mu = -2p_1 A_1^2$. Because $d\mu/d\tau_1 = 0$, the planes $\mu =$ constant are invariant. From eigenvalue analysis, the equilibria at μ axis is asymptotically stable if $\mu < 0$ and unstable if $\mu > 0$. Using similar analysis, the equilibria at $\mu = -2p_1 A_1^2$ are asymptotically stable for $\mu > 0$ and unstable if $\mu < 0$.

The bifurcation diagrams depicting the preceding description are given in Fig. 3. These diagrams show that there is a finite amplitude oscillation of limit-cycle type appearing and dissappearing in the system as μ varies across $\mu = 0$ (Hopf bifurcation). However, only for $p_1 < 0$ the system sustains a stable limit cycle. Physically, this means that only for this situation the sustained wing-rock motion can exist.

In case where the stable limit cycle exists in the system, the amplitude of the limit cycle, that is, of the wing-rock motion in steady state is given by

$$A_1 = \sqrt{-\mu/2p_1}, \qquad A_2 = 0 \tag{49}$$

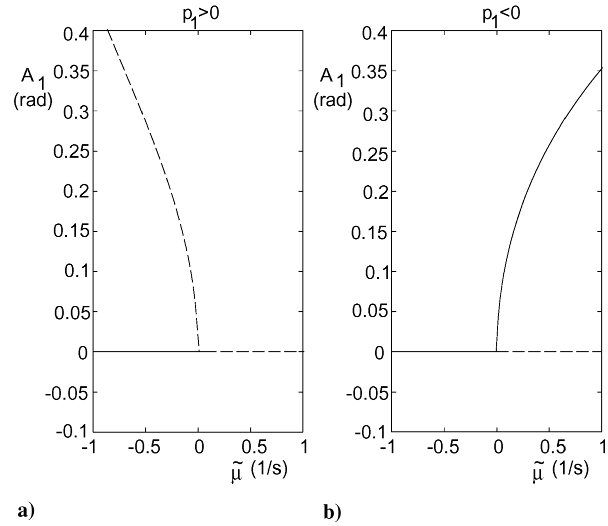

Fig. 3 Bifurcation diagrams for a) $p_1 > 0$ and b) $p_1 < 0$.

The steady-state correction to the phase then can be obtained as follows:

$$B_1 = -p_3(\mu/2p_1)\tau_1, \qquad B_2 = -q_3(\mu/2p_1)\tau_1 \tag{50}$$

C. Analytical Approximation of the Solutions

The analytical solutions of the system are derived from the amplitude and phase-correction equations. Rewriting Eq. (37),

$$A_3\frac{dA_3}{d\tau_2} = \frac{1}{2}\vartheta A_3^2 + \frac{1}{8}c_2 A_3^4 \tag{51}$$

This is integrated to yield

$$A_3 = \frac{\sqrt{K_1\vartheta}\exp[(\vartheta/2)\tau_2]}{\sqrt{1 - \frac{1}{4}K_1 c_2 \exp(\vartheta\tau_2)}} \tag{52}$$

where K_1 is a constant determined from the initial condition.

For A_3 to be meaningful as the departure from the equilibrium condition, then from Eq. (52), $K_1 < 0$ for $\vartheta < 0$ and $K_1 > 0$ for $\vartheta > 0$. Note that the assumption $c_2 > 0$ is embedded in the preceding statement. For $\vartheta < 0$, the numerator of Eq. (52) goes to zero as $\tau_2 \to \infty$. Therefore, $A_3 \to 0$ as $\tau_2 \to \infty$. For $\vartheta > 0$, the denominator of Eq. (52) becomes smaller as τ_2 increases, while the value of the numerator increases. Hence, A_3 increases as τ_2 increases, and at some τ_2, $A_3 \to \infty$. In other words, the solution diverges for $\vartheta > 0$.

From Eq. (25) and invoking Gronwall's lemma,[23] A_2 can be approximated as

$$\bar{A}_2(\tau_2) = A_{2_0}\exp\left(\tfrac{1}{2}\nu\tau_1\right) \tag{53}$$

Substitution to the amplitude equation in Eq. (22) results in

$$\frac{dA_1}{d\tau_2} = a(\tau_2)A_1 + p_1 A_1^3 \tag{54}$$

where

$$a(\tau_1) = \tfrac{1}{2}\mu + p_2 A_{2_0}^2 \exp(\nu\tau_2) \tag{55}$$

The exact solution of Eq. (54) is derived as

$$A_1 = \frac{\exp\left[\int a(\tau_2)\,d\tau_2\right]}{\sqrt{K_2 - 2p_1\int\exp\left[2\int a(\tau_2)\,d\tau_2\right]d\tau_2}} \tag{56}$$

The constant K_2 depends on the initial condition. The integral $\int\exp[2\int a(\tau_2)\,d\tau_2]\,d\tau_2$ is not simple because $a(\tau_2)$ also contains an exponential term. However, for integration limit from 0 to τ_2 the integral has the following properties:

$$\int_0^{\tau_2} \exp\left[2\int_0^{\tau_2} a(\tau_2)\,\mathrm{d}\tau_2\right]\mathrm{d}\tau_2 = 0, \qquad \tau_2 = 0 \tag{57}$$

$$\int_0^{\tau_2} \exp\left[2\int_0^{\tau_2} a(\tau_2)\,\mathrm{d}\tau_2\right]\mathrm{d}\tau_2 \approx \frac{2}{\mu}\exp\left(\frac{\mu}{2}\tau_2\right) \qquad \tau_2 \gg 1 \tag{58}$$

As $\tau_1 \to \infty$,

$$\exp[(\mu/2)\tau_2] \to 0 \qquad \text{for} \qquad \mu < 0$$

$$\exp[(\mu/2)\tau_2] \to \infty \qquad \text{for} \qquad \mu > 0 \tag{59}$$

Based on this, we obtain

$$A_1 \to 0 \qquad \text{for} \quad \mu < 0$$

$$A_1 \to \sqrt{-\frac{\exp(\mu\tau_1)}{2(p_1/\mu)\exp(\mu\tau_1)}} = \sqrt{-\frac{\mu}{2p_1}} \quad \text{for} \quad \mu > 0 \tag{60}$$

IV. Comparison with Numerical Results

A comparison of the analytical prediction with numerical results is shown for a generic fighter aircraft model (Appendix), whose variations of the parameters μ, p_1, and ϑ with the nominal angle of attack are shown in Fig. 4. We see that the onset of wing rock ($\mu = 0$) in this case is $\alpha_0 = 29.23$ deg.

The accuracy of the prediction is first examined by simulating the aircraft response slightly above and below the onset point at $\alpha_0 = 29.1$ and 29.4 deg (Fig. 5). The response is stable for 29.1 deg and shows wing rock for $\alpha_0 = 29.4$ deg, thereby validating the accuracy of the analytical prediction.

For $\alpha_0 = 31$ deg, the analytical result, in comparison with numerical results (Fig. 6), predicts the amplitude history and the limit-cycle frequency very well. The existence of the new equilibrium and sustained oscillation in the longitudinal mode with frequency twice of the lateral motion is correctly predicted by the analytical approximation. Also note that initially the roll motion is not symmetrical about its equilibrium. This asymmetry is caused by the presence of the third mode in the system $C(t)$ (Fig. 7). This mode is slower than the other modes of the aircraft and is also predicted correctly by our analytical method.

The preceding examples demonstrate that our analysis predicts accurately the dynamics of a very complicated aircraft model.

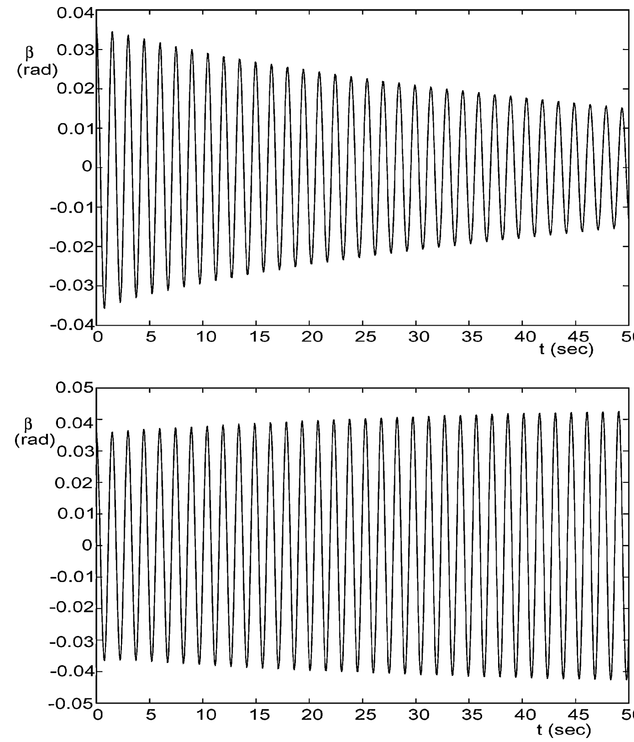

Fig. 5 Aircraft response for $\alpha_0 = 29.1$ and 29.4 deg.

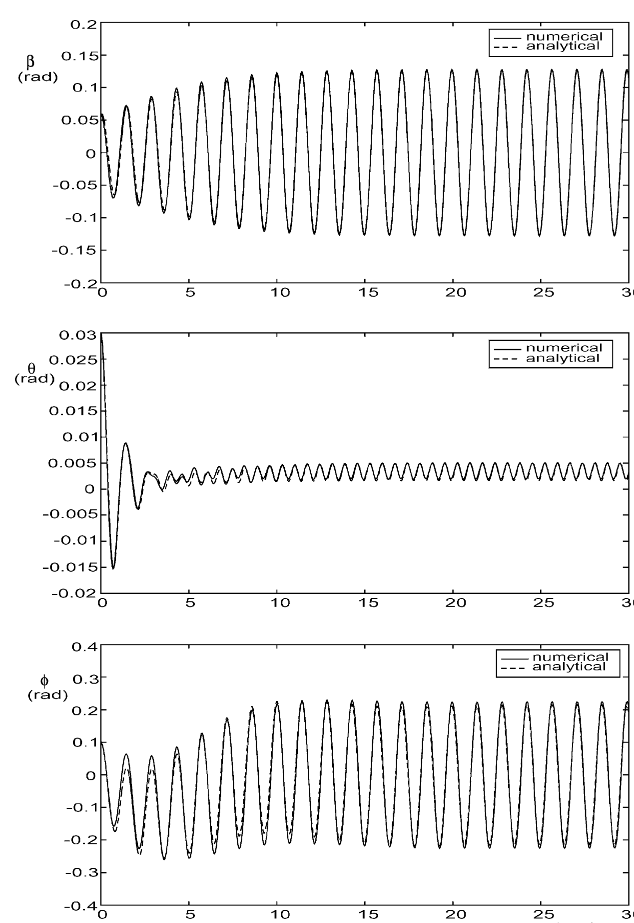

Fig. 6 Aircraft response at $\alpha_0 = 31$ deg and initial condition $(\beta_0, \phi_0, \theta_0) = (0.058, 0.1, 0.03)$ rad.

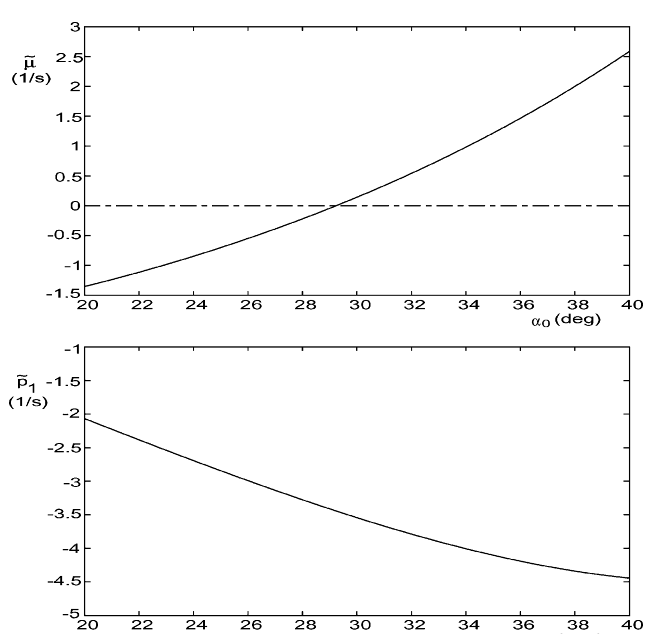

Fig. 4 Variation of $\tilde{\mu}$ and $\tilde{p}_1$ with α_0.

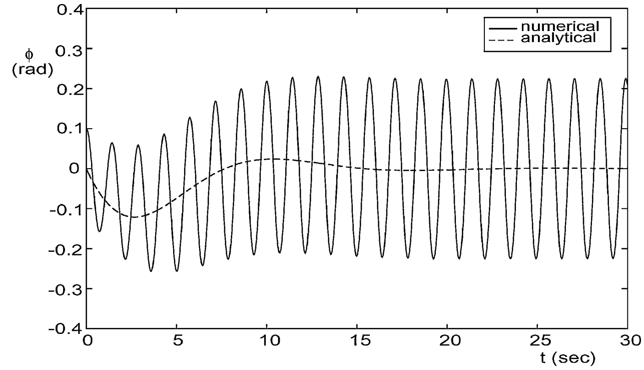

Fig. 7 Third mode of the motion compared to roll response at $\alpha_0 = 31$ deg.

Further, the analysis obtains the solutions in parametric forms, which are very useful in assessing the effects of aircraft parameters on the overall aircraft dynamics, in contrast to the purely numerical approaches.

The effects of certain types of aerodynamic nonlinearities on the system dynamics can be examined individually.[17] One interesting result that is not observed in the lower degree-of-freedom models having roll as the only lateral degree of freedom is the potential of wing rock caused by strong cubic variations of lateral moments with angle of sideslip. It is shown in Ref. 17 that such a nonlinearity does not cause wing rock when it is weak, but it could give rise to wing rock when it is relatively strong. The preceding analysis has to be slightly modified to capture this effect.

V. Conclusions

This work demonstrates the applicability of the analytical methodology combining the Multiple Time Scales method, center manifold reduction techniques, and bifurcation theory to study wing-rock dynamics of an aircraft having three degrees of freedom in roll, pitch, and yaw. This paper shows that such a methodology is able to systematically separate the complex dynamics into simpler fast and slow modes, which leads into considerable insight of the dynamics. Also the methodology yields results in parametric forms, which have been shown to compare very well with the numerical results. The results of the analysis using the methodology show that the wing-rock properties are affected by parameters from all rotational axes. Therefore, for an accurate prediction of wing-rock motion characteristics the three-degree-of-freedom model such as this should be used.

Appendix: Parameters of a Generic Fighter Aircraft for 20 deg ≤ α_0 ≤ 40 deg

$$I_{xx} = 36{,}610 \text{ kg m}^2, \qquad I_{yy} = 162{,}700 \text{ kg m}^2$$

$$I_{zz} = 183{,}000 \text{ kg m}^2, \qquad I_{xz} = 6{,}780 \text{ kg m}^2$$

$$b = 12 \text{ m}, \qquad c = 4.8 \text{ m}, \qquad S = 164.6 \text{ m}^2$$

$$\rho = 1.225 \text{ kg/m}^3, \qquad V = 100 \text{ m/s}$$

$$\begin{aligned} C_l = {} & \left(-1.18\alpha_0 + 0.79\alpha_0^2\right)\beta + 0.4\beta^3 - 0.08\alpha\beta - 0.1\beta p \\ & + \left(-0.22 + 0.63\alpha_0 + 0.797\alpha_0^2 + 0.975\alpha_0^3\right)p - 0.006p^3 \\ & - 1.42\beta^2 p + 0.56\alpha p + 0.09\alpha^2 p + 0.5\beta q - 3\alpha\beta q \\ & - 0.011\dot{\beta} + 1.6\alpha\dot{\beta} - 6.1\alpha^2\dot{\beta} + 0.05r - 0.03\beta^2 r \\ & + 0.1r^3 + 1.43\alpha r + 2.29\alpha^2 r + 0.236\alpha^2\beta \end{aligned}$$

$$\begin{aligned} C_m = {} & -0.68\alpha - 0.75\alpha^2 - 3.75\alpha^3 + 0.1p^2 - 8.02\alpha\beta + 0.26\beta^2 \\ & + 0.1\beta p + 5\alpha\beta p - 2q + 0.58\alpha q + 3.564\alpha^2 q + 0.1q^2 \\ & - 0.5\dot{\alpha} + 0.5\beta r \end{aligned}$$

$$\begin{aligned} C_n = {} & 0.25\beta - 0.19\alpha\beta - 0.7\alpha^2\beta - 0.025\beta^3 + 0.1p + 0.02p^3 \\ & - 0.07\alpha p + 2.8\alpha^2 p - 0.3r - 2\beta^2 r - 0.01r^3 + \alpha r \\ & - 0.1\dot{\beta} - 0.2\beta q + 0.5\alpha\beta q - 3.19(\beta^2 p + \alpha^2 r) \end{aligned}$$

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